Autoresonance of coupled nonlinear waves

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Adiabatic passage of weakly coupled nonlinear waves with space-time varying parameters through resonance is investigated. Slow evolution equations describing this wave interaction problem are obtained via Whitham's averaged variational principle. Autoresonant solutions of these equations are found and, locally, comprise adiabatically varying quasiuniform wave train solutions of the decoupled problem. At the same time, the waves are globally phase locked in an extended region of space-time despite the variation of the system's parameters. Conditions for entering and sustaining this multidimensional autoresonance are the internal resonant excitation of one of the coupled waves and sufficient adiabaticity and nonlinearity of the problem. These conditions have their origin in a similar adiabatic resonance problem in nonlinear dynamics. The theory is illustrated by an example of the autoresonance in a system of coupled sine-Gordon equations.
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I. INTRODUCTION

Resonant interactions of waves with adiabatically varying parameters are of interest in plasma physics, nonlinear optics, acoustics, etc. The simplest example is the linear mode conversion in space-time varying media, taking place when an externally launched eikonal wave passes a region where it resonates with another weakly coupled linear wave. The second wave is excited in the resonant region, but then decouples from the original wave because of the nonuniformity and/or time dependence of the background and the two waves propagate independently outside the resonant region. The linear mode conversion has been studied in the past in the context of waves in ionospheric plasmas [1] and, more recently, is used as one of the principal plasma heating mechanisms [2]. Mathematically more complex than linear mode conversion but, nevertheless, very important in applications, are three-wave resonant interactions in adiabatically varying media [3]. In this case a triad of waves interacts resonantly in a localized region of space-time via a quadratic (in terms of the wave amplitudes) nonlinear coupling term, while, typically, the decoupled waves are viewed as linear. Recently, it was shown that if, in addition to the variation of the parameters, one includes the nonlinear dispersion of the resonantly interacting waves, there exists a possibility of entering a new *autoresonant* regime in the mode conversion and three-wave interaction processes [4]. In autoresonance, the coupled waves tend to stay in resonance in an enlarged region of space-time. The broadening of the resonant region is accompanied by an automatic self-adjustment of the wave amplitudes, allowing the continuation of the phase locking between the waves despite the variation of system's parameters. The autoresonance of nonlinear waves is the space-time generalization of the autoresonance in nonlinear dynamics, where, in early studies, the phenomenon was used in particle accelerators [5] and, recently, in other dynamical problems [6], as well as in the atomic [7] and plasma physics [8]. The autoresonance is well understood in general dynamical systems with one degree of freedom. Nevertheless, the wave autoresonance theory of Ref. [4] was limited to studying *weakly* nonlinear cases only.

Recently, a further theoretical progress was achieved in showing the existence of the autoresonant solutions for a large class of driven, *fully* nonlinear waves [9]. A typical example is the driven sine-Gordon equation (the case considered in detail in Ref. [9])

$$u_{tt}-c^2u_{xx}+\omega_0^2 \sin u=\varepsilon v, \tag{1}$$

where c and ω_0 may vary adiabatically, $\varepsilon \ll 1$ is a small dimensionless coupling parameter, and $v=b\cos(\psi+\psi_0)$ is a *given* eikonal driving (pump) wave and $b(x,t)$, $\psi(x,t)$ and $\psi_0(x,t)$ are the amplitude, phase, and phase modulation of the pump, respectively. One also assumes that b, the frequency $\omega(x,t)\equiv-\psi_t$, the wave vector $k(x,t)\equiv\psi_x$, and ψ_0 are adiabatically varying functions of space-time. It was shown in Ref. [9] that, under certain conditions, the autoresonance in this driven system proceeds as the pump wave v passes the space-time region [a strip around a line in the (x,t) plane] where it resonates with the linear daughter wave described by the left-hand side of Eq. (1). The theory of Ref. [9] was not limited to Eq. (1), but described a larger class of driven multidimensional nonlinear systems given by the variational principle

$$\delta_u\left(\int\int \mathrm{L}\, dx\, dt\right)=0, \tag{2}$$

where the Lagrangian was of the form

$$\mathrm{L}=L(u_t,u_x,u,q)+\varepsilon v u, \tag{3}$$

and $q(x,t)$ represented a set of adiabatically varying parameters, while v was the aforementioned eikonal pump wave. Lagrangian (3) yields the following variational evolution equation for the daughter wave $u(x,t)$:

$$\partial_t(L_{u_t})+\partial_x(L_{u_x})-L_u=\varepsilon b\cos\psi. \tag{4}$$

The progress in the theory of the autoresonance in systems described by Eq. (4) was achieved by developing the aver-

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aged variational principle for studying the autoresonant wave interactions [9]. Nevertheless, this theory was still limited to the prescribed pump wave case only.

In the present work we shall include the self-consistent autoresonant evolution of the pump wave. We shall assume that the complete pump-daughter wave system is described by the variational principle (2), but now v is a dependent field variable, and

$$\mathrm{L}=L(u_t,u_x,u,q)+M(v_t,v_x,v,r)+\varepsilon uv. \tag{5}$$

Here $M(v_t,v_x,v,r)$ is the Lagrangian of the unperturbed pump wave, $r(x,t)$ is a slow parameter, and the interaction term εuv describes a weak linear coupling. Lagrangian (5) yields the following coupled evolution equations:

$$\begin{aligned}\partial_t(L_{u_t})+\partial_x(L_{u_x})-\mathrm{L}_u&=\varepsilon v,\\ \partial_t(M_{v_t})+\partial_x(M_{v_x})-M_v&=\varepsilon u.\end{aligned} \tag{6}$$

Our goal is to find autoresonant solutions in this system, i.e., a situation when an externally launched pump wave excites the daughter wave as they resonate inside the space-time region of interest, while after the excitation the waves propagate phase locked in an extended region of space-time. The self-consistent description of the evolution of the waves in the autoresonance and the study of the stability of the autoresonant interaction are the main targets of this work. Our presentation will be as follows. In Sec. II we shall apply the averaged variational principle idea in deriving the slow evolution equations for studying the autoresonance in the coupled phase locked daughter-pump wave system. The boundary conditions and the process of trapping into the resonance (the initial excitation stage) will also be discussed in Sec. II. Section III will deal with the autoresonant solutions of the slow evolution equations. An application of our theory to the case of coupled sine-Gordon equations and numerical examples will be presented in Sec. IV. Finally, Sec. V gives our conclusions.

II. SLOW EQUATIONS FOR PHASE-LOCKED WAVES

The averaged variational principle for studying slow modulations of nonlinear waves was developed by Whitham [10], and Ref. [9] comprised an application of a similar approach to driven autoresonant waves. Many details of the developments for studying the autoresonant evolution of the complete daughter-pump wave system are similar to those of Ref. [9]. Thus we shall skip these details in the present theory. We proceed by introducing the two-scale representations [9,10] of the solutions of Eq. (6), i.e., $u(x,t)=U[\theta(x,t),X,T]$ and $v(x,t)=V[\psi(x,t),X,T]$, where $X\equiv\varepsilon x$ and $T\equiv\varepsilon t$ (ε is viewed as the largest small parameter in the problem), and we assume the periodicity of U and V with respect to the fast phase variables θ and ψ. The rationale beyond these representations is the expectation of having solutions of the coupled problem which, locally (on the fast variation scale), comprise quasiuniform wave train solutions for the decoupled waves, but, at the same time, involve adiabatic modulations on the slow scale (X,T) because of the presence of the weak coupling and the adiabatic variations of the system's parameters. By expanding in Fourier series, $U=\Sigma_n a_n\exp(in\theta)$ and $V=\Sigma_n b_n\exp(in\psi)$, substituting these expansions into the right-hand side of Eqs. (6), and leaving only the $n=1$ terms for simplicity, i.e., focusing on the fundamental resonance problem, we obtain

$$\begin{aligned}\partial_t(L_{u_t})+\partial_x(L_{u_x})-L_u&=\varepsilon b\cos(\psi+\psi_0),\\ \partial_t(M_{v_t})+\partial_x(M_{v_x})-M_v&=\varepsilon a\cos(\theta+\theta_0),\end{aligned} \tag{7}$$

where $a,\ b=|a_1|,|b_1|$, and $\theta_0,\psi_0=\mathrm{Arg}(a_1,b_1)$. Note that each of the equations in Eq. (7) has the same form as Eq. (4) studied in Ref. [9]. Therefore, the solution of the coupled system can proceed along the lines similar to those outlined in that work.

We seek globally phase locked solutions for the coupled waves, i.e., write $\theta=\phi(x,t)+\Theta(X,T)$ and $\psi=\phi(x,t)+\Psi(X,T)=\theta-\Theta+\Psi$, where ϕ is the rapidly varying part of the phases (assumed to be *the same* for both waves, which is the phase locking assumption), while Θ and Ψ represent *bounded* modulations. We shall also assume that the frequency $\omega=-\phi_t$ and the wave vector $k=\phi_x$ associated with ϕ are slowly varying and smooth, while Θ and Ψ oscillate and scale with x and t as $\Theta=\Theta(\varepsilon^{1/2}x,\varepsilon^{1/2}t)$ and $\Psi=\Psi(\varepsilon^{1/2}x,\varepsilon^{1/2}t)$ (see below). Note, at this stage, that we have introduced three phases ϕ, Θ, and Ψ, instead of the two original angle variables θ and ψ. Therefore, we have the freedom of adding an additional constraint. We shall use this freedom later. Now, we further exploit the similarity between the equations in Eq. (7) in the self-consistent problem, and Eq. (4) for the prescribed pump case. The latter was studied in Ref. [9] via the averaged variational principle, and here, for completeness, we present a brief derivation of the slow autoresonant evolution equations in this theory. The idea was based on the possibility of replacing the original principle (2) by an equivalent variational principle [10]

$$\delta\left(\int\int\langle\mathrm{L}\rangle dX\,dT\right)=0, \tag{8}$$

where $\langle\mathrm{L}\rangle\equiv(2\pi)^{-1}\int_0^{2\pi}\mathrm{L}\,d\theta$ is the Lagrangian averaged over one period of the fast phase variable, holding the slow variables X and T fixed. To first order in ε,

$$\langle\mathrm{L}\rangle=L^0+\varepsilon ab\cos(\Theta+\theta_0-\psi_0)+\alpha, \tag{9}$$

where the zero order averaged Lagrangian

$$L^0=L^0(A,k+\Theta_x,\omega-\Theta_t,q) \tag{10}$$

is calculated by using the uniform wave train solutions of the unperturbed nonlinear wave problem with constant parameters having the same values as in our exact problem at given X and T. Note that k and ω in Eq. (10) are those of the pump wave, while A is the energy variable of the wave train [9]. The small term α in Eq. (9) involves space-time derivatives of the slow parameters of the wave train solution and need not be specified. An approximate form of Eq. (10), $L^0\approx L^0(A,k,\omega,q)+L_k^0(A,k,\omega,q)\Theta_x-L_k^0(A,k,\omega,q)\Theta_t$, was used in Ref. [9] in the variational principle. However, because of the above-mentioned special scaling of Θ with x and t, we have $\Theta_{x,t}\sim O(\varepsilon^{1/2})$ and, to $O(\varepsilon)$, we must also include the terms with Θ_x^2,Θ_t^2 and $\Theta_x\Theta_t$ in approximating

L^0 in the variational principle. Alternatively, in this work, we shall postpone the expansion until analyzing the stability of the variational evolution equations (see Sec. III). This is an important improvement of the previous theory. The averaged Lagrangian (9) is a function of *slow* variables and parameters A, Θ, k, ω, and q only. Therefore, by taking the variations in Eq. (8) with respect to the dependent field variables A and Θ, we obtain the following system of slow evolution equations:

$$L^0_{\omega t} - L^0_{kx} = -\varepsilon ab \sin\Phi, \tag{11}$$

$$L^0_A = \varepsilon a_A b \cos\Phi + \gamma, \tag{12}$$

where $\Phi = \Theta + \theta_0 - \psi_0$, while γ [similarly to α in Eq. (9)] is a small function involving the space-time derivatives of various slow parameters in the problem, and plays a negligible role in the autoresonance [9] and, thus, will be omitted in the following. Finally, note that L^0_ω and $-L^0_k$ in Eqs. (11) and (12) are the action and the action flux densities of the wave train solution for the daughter wave [10], but one evaluates these objects at slightly *shifted* local ω and k values associated with the pump.

Now we generalize the theory to include the self-consistent evolution of the pump wave. As mentioned above, each of the two equations in Eq. (7) has a form similar to Eq. (4). On the other hand, to $O(\varepsilon)$, Eq. (4) can be replaced by a pair of slow evolution equations (11) and (12). Therefore, to the same order, Eqs. (7) for the self-consistent problem can be replaced by a system of two pairs of slow evolution equations [compare to Eqs. (11) and (12)]

$$L^0_{\omega t} - L^0_{kx} = -\varepsilon ab \sin\Phi, \tag{13}$$

$$L^0_A = \varepsilon a_A b \cos\Phi, \tag{14}$$

$$M^0_{\omega t} - M^0_{kx} = \varepsilon ab \sin\Phi, \tag{15}$$

$$M^0_B = \varepsilon a b_B \cos\Phi, \tag{16}$$

where $\Phi \equiv \Theta - \Psi$, while $L^0 = L^0(A, k+\Theta_x, \omega - \Theta_t, q)$ and $M^0 = M^0(B, k+\Psi_x, \omega - \Psi_t, r)$ are the averaged Lagrangians of the decoupled daughter and pump waves, respectively, and B is the energy variable (the analog of A) parametrizing the wave train solutions for the decoupled pump wave. Furthermore, we have omitted the irrelevant small terms [the analogs of γ in Eq. (12)] in the right-hand side of Eqs. (14) and (16) (see the remark in the previous paragraph) and included, at this point, θ_0 and ψ_0 in the definitions of Θ and Ψ. Finally, we observe that Eqs. (13) and (15) yield the conservation law (the Manley-Rowe relation)

$$(L^0_\omega + M^0_\omega)_t - (L^0_k + M^0_k)_x = 0. \tag{17}$$

At this stage, we discuss the boundary conditions. The latter must be consistent with the slow evolution and phase locking assumptions. We shall be focusing on the *internal* daughter wave excitation problem, i.e., on the situation when a large amplitude pump wave is launched at the boundary of the region of interest, while the amplitude a of the daughter wave on the boundary is negligible. Then a will remain small until the daughter wave resonates with the pump wave. Consequently, in the region between the boundary and until the pump wave reaches the resonance (the *initial excitation region* in the following) one can neglect the interaction terms in Eqs. (15) and (16) for the pump wave, i.e., write

$$M^0_{\omega t} - M^0_{kx} = 0, \tag{18}$$

$$M^0_B = 0. \tag{19}$$

Next, we use the above-mentioned freedom of prescribing one of the phases ϕ, Ψ or Θ, and set $\Psi = \text{const}$ in the initial excitation region, so, in Eqs. (18) and (19),

$$M^0 = M^0(B, k, \omega, r). \tag{20}$$

One can identify Eqs. (18) and (19) with the usual system of slow evolution equations for the energy density B and phase ϕ of the adiabatically varying wave train solution for the pump wave [10]. This is a pair of coupled partial differential equations (PDE's), and we shall assume that the solutions $B = B(x,t)$, and $\phi = \phi(x,t)$ [and, therefore, also $k = k(x,t)$ and $\omega = \omega(x,t)$] of these equations are known. In other words, the pump wave is *prescribed* in the entire initial excitation region.

Now we discuss the initial excitation of the daughter wave. The latter is described by Eqs. (13) and (14), where, of course, one cannot neglect the coupling term. Nevertheless, if one starts at the boundary, where the amplitude of the daughter wave is small by assumption, one can use the above-mentioned decoupled pump wave solution in solving Eqs. (13) and (14) in the initial excitation region, so the initial excitation stage of the daughter wave reduces to that studied in Ref. [9]. It was shown in that work that the solution for the daughter wave in this region is as follows. Starting at the boundary, the driven daughter wave enters the strong phase trapping stage, in which its amplitude is still small, but the phase difference $\Phi = \Theta - \Psi \pmod{2\pi}$ becomes near either 0 or π. This phase locked small amplitude (linear) driven wave propagates until it reaches the resonance line in the (x,t) plane determined by the equation

$$D[\omega(x,t), k(x,t), q(x,t)] \sim L^0_A = 0. \tag{21}$$

Here D is the linear dispersion function characterizing the small amplitude daughter wave in the decoupled problem, and is evaluated at local values of the frequency and wave vector of the adiabatically varying pump wave. Thus, indeed, the daughter and pump waves *resonate* on the line given by Eq. (21). In the vicinity of this resonance line an efficient excitation of the daughter wave takes place and, under certain conditions, the daughter wave may enter the autoresonant interaction stage [9]. As the amplitude of this wave increases beyond the resonance line, one may also expect a violation of the prescribed pump wave assumption. Our next goal is to remove this assumption, i.e., to consider the full system of Eqs. (13)–(16) in the autoresonant interaction stage. Note that the amplitudes and phases of the two waves are already known on the resonance line, so one can view this line as a *new boundary* in the problem. Also, beyond the linear resonance line, the daughter wave amplitude is large, and one can remove the $O(\varepsilon)$ interaction terms in Eqs. (14) and (16) in studying the autoresonant evolution stage.

III. AUTORESONANT SOLUTIONS

In this section, we seek autoresonant-type solutions of the evolution equations (13)–(16) beyond the resonance line, i.e., write $(A,B,\Theta,\Psi)=(\overline{A},\overline{B},\overline{\Theta},\overline{\Psi})+(\delta A,\delta B,\delta\Theta,\delta\Psi)$, splitting the *smooth* averages from *small* oscillating components. By separating the averaged and the oscillating parts in Eqs. (13)–(16), we have

$$(\overline{L^0_{\omega A}}\partial_t-\overline{L^0_{kA}}\partial_x)\overline{A}=\beta_a-\varepsilon\overline{a\bar{b}}\,\Phi, \tag{22}$$

$$(\overline{M^0_{\omega B}}\partial_t-\overline{M^0_{kB}}\partial_x)\overline{B}=\beta_b+\varepsilon\overline{\bar{a}b}\,\Phi, \tag{23}$$

$$(\overline{L^0_{\omega A}}\partial_t-\overline{L^0_{kA}}\partial_x)\overline{\Theta}\approx\overline{L^0_A}, \tag{24}$$

$$(\overline{M^0_{\omega B}}\partial_t-\overline{M^0_{kB}}\partial_x)\overline{\Psi}\approx M^0_B \tag{25}$$

and

$$(\overline{L^0_{\omega A}}\partial_t-\overline{L^0_{kA}}\partial_x)\delta A-(\overline{L^0_{\omega\omega}}\partial_t^2+\overline{L^0_{kk}}\partial_x^2-2\overline{L^0_{\omega k}}\partial_t\partial_x)\delta\Theta$$
$$=-\varepsilon\overline{a\bar{b}}(\delta\Theta-\delta\Psi), \tag{26}$$

$$(\overline{M^0_{\omega B}}\partial_t-\overline{M^0_{kB}}\partial_x)\delta B-(\overline{M^0_{\omega\omega}}\partial_t^2+\overline{M^0_{kk}}\partial_x^2-2\overline{M^0_{\omega k}}\partial_t\partial_x)\delta\Psi$$
$$=\varepsilon\overline{\bar{a}b}(\delta\Theta-\delta\Psi), \tag{27}$$

$$(\overline{L^0_{\omega A}}\partial_t-\overline{L^0_{kA}}\partial_x)\delta\Theta\approx\overline{L^0_{AA}}\delta A, \tag{28}$$

$$(\overline{M^0_{\omega B}}\partial_t-\overline{M^0_{kB}}\partial_x)\delta\Psi\approx\overline{M^0_{BB}}\delta B, \tag{29}$$

where $\overline{(\cdots)}$ means evaluations at ω, k, $\overline{A}$, and $\overline{B}$; the average phase difference $\Phi=\overline{\Theta}-\overline{\Psi}$ is assumed to be small, and $\beta_a\equiv\overline{L^0_{kx}}-\overline{L^0_{\omega t}}$ and $\beta_b\equiv\overline{M^0_{kx}}-\overline{M^0_{\omega t}}$. Next, we use the freedom of choosing one additional constraint on the variables ϕ, Θ, and Ψ beyond the resonance line. We recall that Ψ is constant on the resonance line (the new boundary in the problem), and impose the constancy of $\overline{\Psi}$ in the autoresonant region, so, from Eq. (25),

$$\overline{M^0_B}\equiv 0. \tag{30}$$

This relation can be viewed as an algebraic equation defining $\overline{B}=\overline{B}(\omega,k)$. Furthermore, to $O(\varepsilon)$, Eq. (24) yields

$$\overline{L^0_A}\approx 0, \tag{31}$$

which gives the lowest order approximation for $\overline{A}=\overline{A}(\omega,k)$. On the other hand, by adding Eqs. (22) and (23), one obtains the averaged Manley-Rowe relation

$$(\overline{L^0_{\omega A}}\partial_t-\overline{L^0_{kA}}\partial_x)\overline{A}+(\overline{M^0_{\omega}}\partial_t-\overline{M^0_{kB}}\partial_x)\overline{B}=\beta_a+\beta_b\,. \tag{32}$$

This equation, upon substitution of $\overline{A}(\omega,k)$ and $\overline{B}(\omega,k)$ determined above, can be viewed as a PDE for ω and k, and, in combination with the consistency condition

$$\partial_t k+\partial_x\omega=0, \tag{33}$$

yields a complete system of two first order PDE's for slowly varying ω and k in our problem. The solution of this system requires integration along the characteristics originating on the boundary of the autoresonant region (the linear resonance line). After finding ω and k we determine $\overline{A}$ and $\overline{B}$ via (30) and (31). These solutions can be also used in checking the assumption of the smallness of Φ, via, say, Eq. (22). Finally, we observe that in contrast to the multidimensional case, finding the averaged energy densities $\overline{A}$ and $\overline{B}$ in one-dimensional situations involves solutions of algebraic equations only. Indeed, suppose one treats a stationary (constant ω) problem, where the slow parameters q and r depend on x only. Then Eq. (17) yields, $L^0_k+M^0_k=\text{const}$. The part of this equation averaged over the autoresonant oscillations in combination with Eqs. (30) and (31) comprise a set of algebraic equations for $\overline{A}$, $\overline{B}$, and k as functions of x.

As the final step in our theory, we proceed to the oscillating autoresonant components described by Eqs. (26)–(29). Since, at this stage, the adiabatic averages $\overline{A}$ and $\overline{B}$ are already known, these equations comprise a set of homogeneous, linear, first order PDE's with slowly varying coefficients. Therefore, the oscillating components can be found by using the usual multidimensional WKB approximation [11]. We shall not describe the details of this method here, and focus only on finding the frequency ν and the wave vector κ of the autoresonant oscillations. To lowest order in the WKB approximation, Eqs. (26)–(29) become

$$\begin{aligned}
iS^a\delta A-C^a\delta\Theta&=\varepsilon\overline{\bar{a}b}(\delta\Theta-\delta\Psi),\\
iS^b\delta B-C^b\delta\Psi&=-\varepsilon\overline{\bar{a}b}(\delta\Theta-\delta\Psi),\\
iS^a\delta\Theta+R^a\delta A&=0,\\
iS^b\delta\Psi+R^b\delta B&=0,
\end{aligned} \tag{34}$$

where $S^a=\nu\overline{L^0_{\omega A}}+\kappa\overline{L^0_{kA}}$, $S^b=\nu\overline{M^0_{\omega B}}+\kappa\overline{M^0_{kB}}$, $C^a\equiv\overline{L^0_{\omega\omega}}\nu^2+\overline{L^0_{kk}}\kappa^2+2\nu\kappa\overline{L^0_{\omega k}}$, $C^b\equiv\overline{M^0_{\omega\omega}}\nu^2+\overline{M^0_{kk}}\kappa^2+2\nu\kappa\overline{M^0_{\omega k}}$, $R^a=\overline{L^0_{AA}}$, and $R^b=\overline{M^0_{BB}}$. System (34) yields the local dispersion relation for ν and κ,

$$D^aD^b+\varepsilon\overline{\bar{a}b}(D^a+D^b)=0, \tag{35}$$

where $D^{a,b}\equiv C^{a,b}-(S^{a,b})^2/R^{a,b}$. One can use this relation for evaluating ν and κ in the region of interest by integrating along the characteristics (the rays of the WKB theory) originating on the linear resonance line. Note that the dispersion relation (35) is real, yielding real ray equations and, in turn, real solutions for ν and κ in the parts of the autoresonant region accessible by the rays. This guarantees the stability of the autoresonant oscillations δA, δB, $\delta\Theta$, and $\delta\Psi$ in the accessible region. Note that our stability analysis simplifies significantly in one-dimensional, stationary (constant ω) problems (we have discussed the question of finding $\overline{A}$ and $\overline{B}$ in this situation earlier). In these problems, we seek time independent solutions for the oscillating autoresonant components and, thus, set $\nu=0$ in Eq. (35), yielding a simple stability condition $\varepsilon\overline{\bar{a}b}(d^ad^b)^{-1}(d^a+d^b)>0$, where $d^a\equiv\overline{L^0_{kk}}-(\overline{L^0_{kA}})^2/\overline{L^0_{AA}}$ and $d^b\equiv\overline{M^0_{kk}}-(\overline{M^0_{kB}})^2/\overline{M^0_{BB}}$. Also, in stationary problems, our original system of slow evolution equations reduces to a set of ordinary differential equations

(ODE's), and we shall use this simplified system in the numerical applications in Sec. IV.

Finally, we discuss the validity conditions for our approximations. We observe that, if one denotes by σ ($\sigma \ll 1$) the dimensionless adiabaticity parameter characterizing the space-time variation of r and q in the problem, then $\beta_{a,b}$ in Eq. (32) are of $O(\sigma)$. Then Eq. (32) shows that the adiabaticity parameter characterizing ω and k and, in turn, A and B are also of $O(\sigma)$. As the result, Eq. (22) [or Eq. (23)] yields $\Phi \sim O(\sigma/\varepsilon)$. Therefore, our assumption $\Phi \ll 1$ requires

$$\sigma/\varepsilon \ll 1. \tag{36}$$

On the other hand, according to Eq. (35), ν and κ scale as

$$\nu, k \sim O(\varepsilon^{1/2}), \tag{37}$$

(this is the characteristic autoresonant scaling mentioned above). Then, from Eqs. (34),

$$\Delta A \sim (S^a/R^a)\,\delta\Theta \sim \varepsilon^{1/2}, \quad \delta B \sim (S^b/R^b)\,\delta\Psi \sim \varepsilon^{1/2}, \tag{38}$$

and the smallness of ε appears to guarantee the validity of our first order expansions in powers of δA, and δB, i.e., the validity of

$$\delta A/A \sim S^a/(AR^a) \ll 1, \quad \delta B/B \sim S^b/(BR^b) \ll 1. \tag{39}$$

Nevertheless, these conditions also require a sufficient nonlinearity. Indeed, the functions $S^{a,b}$ in Eq. (39) involve first derivatives of the Lagrangians with respect to the energy densities A and B, while $R^{a,b}$ are the second derivatives of the Lagrangians. But, in linear problems, the Lagrangian is proportional to the energy density, so $R^{a,b}$ vanish and one cannot satisfy Eq. (39). Therefore, in the autoresonance, at least one of the waves must be sufficiently nonlinear. This completes our discussion of the self-consistent autoresonant evolution and the stability of the complete pump-daughter wave system.

IV. EXAMPLE: WEAKLY COUPLED SINE-GORDON EQUATIONS

In this section we illustrate our theory by studying a one-dimensional example of the autoresonant evolution of the solutions of two coupled sine-Gordon equations

$$u_{tt} - c^2 u_{xx} + \omega_a^2 \sin u = \varepsilon v, \tag{40}$$

$$v_{tt} - c^2 v_{xx} + \omega_b^2(x) \sin v = \varepsilon u, \tag{41}$$

where ω_a is constant, while $\omega_b(x)$ is a slowly varying function of position. We shall consider the boundary value problem in which only the pump wave v is excited at some point x_1 and has a form of a wave train $v(x_1,t) = V[\psi(x_1,t), B(x_1)]$ with a given time dependence $\psi(x_1,t) = C - \omega t$ and energy parameter $B(x_1)$. The daughter wave, in contrast, is assumed to be negligible at x_1. The slow evolution equations (13)–(16), in this one-dimensional problem, become

$$L^0_{kx} = \varepsilon ab \sin\Phi, \tag{42}$$

$$L^0_A = \varepsilon a_A b \cos\Phi, \tag{43}$$

$$M^0_{kx} = -\varepsilon ab \sin\Phi, \tag{44}$$

$$M^0_B = \varepsilon a b_a \cos\Phi, \tag{45}$$

where, for the sine-Gordon case [9],

$$L^0 = (\omega^2 - c^2 k_a^2)^{1/2} J^a - A, \quad M^0 = (\omega^2 - c^2 k_b^2)^{1/2} J^b - B, \tag{46}$$

$k_a \equiv k + \Theta_x$, $k_b \equiv k + \Psi_x$ and

$$J^{a,b} = 8\omega_{a,b}\pi^{-1}[E(\pi/2,\kappa^{a,b}) - (1-\kappa^{a,b})F(\pi/2,\kappa^{a,b})], \tag{47}$$

formally, are the actions of the nonlinear oscillators of energies A and B described by the equations $u_{tt} = \omega_a^2 \sin u$ and $u_{tt} = \omega_b^2 \sin u$, respectively [12]. Functions E and F in Eq. (47) are elliptic integrals of the first and second kind, while $\kappa^a = 0.5(1 + A/\omega_a^2)$ and $\kappa^b = 0.5(1 + B/\omega_b^2)$. Furthermore [9], in Eqs. (42)–(45),

$$(a,b) = 4g^{1/2}(1+g)^{-1}, \tag{48}$$

where $g = \exp[-(\pi F'/F)]$ and $F' = F(\pi/2; 1-\kappa)$, and one substitutes $\kappa = \kappa^a$ or κ^b in evaluating a or b, respectively. The parameters $\kappa^{a,b}$ ($0 \leq \kappa^{a,b} \leq 1$) characterize the degree of the nonlinearity of the daughter and the pump waves [9], and small κ values correspond to the linear case (where the wave amplitude is $2\kappa^{1/2}$), while, as $\kappa \to 1$, one approaches the solitary wave solution of the sine-Gordon equation.

Next, we observe that the algebraic equations (43) and (45) allow one to express k_a and k_b via A, B, and $\cos\Phi$, i.e., we write explicit relations $k_{a,b} = G^{a,b}(A,B,\Phi,x)$, where the *slow* x dependence enters because of $\omega_b = \omega_b(x)$. Then, since, formally, $L^0 = L^0(k_a, A)$ and $M^0 = M^0[k_b, B, \omega_b(x)]$, one can rewrite Eqs. (42) and (44) as

$$(L^0_{kA} + L^0_{kk}G^a_A)A_x + L^0_{kk}(G^a_B B_x + G^a_\Phi \Phi_x + G^a_x) = \varepsilon ab \sin\Phi, \tag{49}$$

$$(M^0_{kA} + M^0_{kk}G^b_B)B_x + M^0_{kk}(G^b_A A_x + G^b_\Phi \Phi_x + G^b_x) + M^0_{kx} = -\varepsilon ab \sin\Phi. \tag{50}$$

These two equations, in combination with

$$\Phi_x = k_a - k_b = G^a(A,B,\Phi,x) - G^b(A,B,\Phi,x), \tag{51}$$

comprise a complete set of ODE's for A, B, and Φ.

Now we proceed to our numerical examples. First, we illustrate the possibility of excitation of spatially autoresonant solution for the daughter wave in the case when one neglects the nonlinearity and the coupling in treating the pump wave, i.e., replaces the system of equations (40) and (41) by

$$u_{tt} - c^2 u_{xx} + \omega_a^2 \sin u = \varepsilon v, \tag{52}$$

$$v_{tt} - c^2 v_{xx} + \omega_b^2(x) v = 0. \tag{53}$$

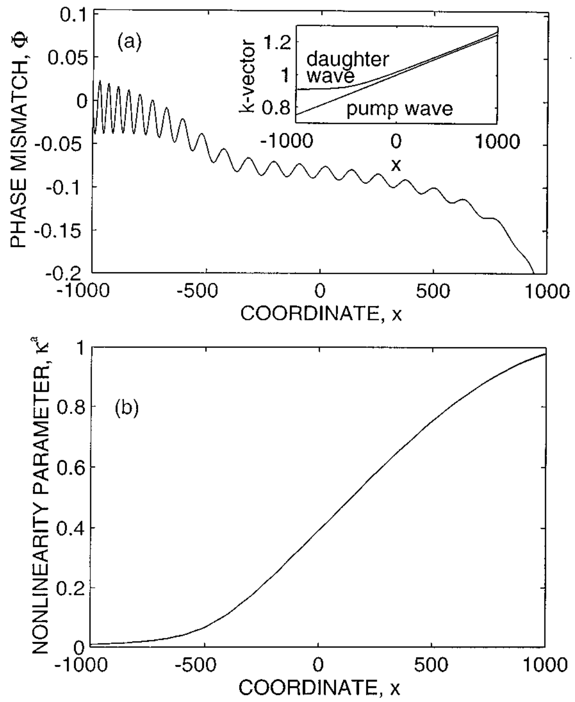

FIG. 1. Spatial autoresonance in the solution of the sine-Gordon equation driven by a prescribed eikonal pump wave. (a) The phase mismatch and the wave vectors of the pump and the daughter waves vs x. (b) The nonlinearity parameters κ^a (the adiabatic theory) and κ'^a [the direct numerical solution of Eq. (52)] of the daughter wave vs x. The two parameters are indistinguishable within the thickness of the line in the figure. The transition $\kappa^a \rightarrow 1$ at large x indicates the approach to the square wave solution.

The system of slow equations (42) and (49) for this case is

$$(L^0_{kA} + L^0_{kk} G^a_A) A_x + L^0_{kk} (G^a_\Phi \Phi_x + G^a_x) = \varepsilon ab \sin \Phi, \tag{54}$$

$$\Phi_x = G^a(A, \Phi, x) - k_b, \tag{55}$$

and the explicit slow x dependence of $G^a(A,\Phi,x)$ is due to $b = b(x)$. This system must be solved in combination with

$$M^0_k = \text{const}(x), \tag{56}$$

$$M^0_B = 0. \tag{57}$$

Recall that the pump wave is linear, by assumption, and, therefore, Eqs. (56) and (57) describe an eikonal wave $v = b(x) \cos[\psi(x) - \omega t]$ of amplitude b and wave vector $k = \partial \psi / \partial x$ satisfying the local dispersion relation $c^2 k_b^2(x) = \omega^2 - \omega_b^2(x)$ [this relation is equivalent to Eq. (57)] and the action flux conservation law $kb^2 = \text{const}(x)$ [the linear limit of Eq. (56)]. Thus we have all the necessary information on the pump wave for solving Eqs. (54) and (55). Figures 1(a) and 1(b) present the results of the numerical solutions of these equations for the phase mismatch Φ and the nonlinearity parameter κ^a of the daughter wave in the cases $c = 1$, $\omega = 1.35$, $\omega_a = 1$, $\omega_b^2 = \omega^2 - (1 + \alpha x)^2$ ($\alpha = 2.5 \times 10^{-4}$), and $\varepsilon = 0.05$, and boundary conditions (at $x_1 = -1000$) $b(x_1) = 1$, $A(x_1) = -0.981$, and $\Phi(x_1) = 0$. One can see in Fig. 1(a) that beyond the initial excitation region ($x_1 < x < -500$) the system settles in the spatially autoresonant regime, where the phase mismatch remains small and oscillates (these are the characteristic stable autoresonant oscillations described above) around a slowly varying average value. The autoresonance effect can be also illustrated by comparing the wave vector of the pump wave ($k = 1 + \alpha x$) to that of the nonlinear daughter wave given by the local dispersion relation $L^0_A = 0$. We make this comparison in the small frame in Fig. 1(a). One observes that beyond $x \approx -500$, the wave vectors of the two waves remain almost the same, i.e., the amplitude of the nonlinear daughter wave self-adjusts so that the wave stays in an approximate resonance with the pump. Finally, the approach of κ^b to unity indicates the transition to the limiting square wave solution for the daughter wave as x increases. In order to test our adiabatic theory we also performed direct numerical solutions of Eq. (52) with the prescribed eikonal pump wave. We used a standard spectral method [13] in our numerical tests, and confirmed the accuracy of the method by doubling the number of harmonics used in the calculations and by reducing twice the spatial integration step. We used matched boundary condition for u, i.e., $u(x_1,t) = \varepsilon v(x_1,t)/[\omega^2 - c^2 k^2(x_1) - \omega_a^2]$. This boundary condition guarantees smooth excitation of the daughter wave and corresponds to the case of a vanishing solution if one moves further away from the linear resonance line, i.e., when $|\omega^2 - c^2 k^2(x_1) - \omega_a^2| \rightarrow \infty$. The results of our direct numerical solution of Eq. (52) with this boundary condition and for the same parameters as for the slow equations above are also presented in Fig. 1(b). In addition to $\kappa^a = 0.5(1 + A/\omega_a^2)$, in this figure, we show $\kappa'^a = 0.5(1 + A'/\omega_a^2)$, where the numerically evaluated function $A' \equiv \langle 0.5(u_t^2 - c^2 u_x^2) - \omega_a^2 \cos u \rangle_{av}$ is used with the averaging taken over one temporal oscillation, i.e., during the period $2\pi/\omega$. This function corresponds to the energy variable A in the adiabatic theory. Importantly, A' and A are *indistinguishable* within the line thickness in Fig. 1(b), illustrating the accuracy of our adiabatic theory.

At this point, we include the nonlinearity of the pump and its self-consistent evolution due to the coupling with the daughter wave, i.e., we consider the full system of the slow evolution equations (49), (50) and (51). Figures 2(a) and 2(b) present the results obtained by solving these equations numerically. We choose some of the relevant parameters in this example as in Fig. 1, i.e., $c = 1$, $\omega_a = 1$, $\omega_b^2 = \omega^2 - (1 + \alpha x)^2$ ($\alpha = 2.5\ 10^{-4}$), and $\varepsilon = 0.05$, but $\omega = 1.44$ and the boundary conditions (at $x_1 = -1000$) are $A(x_1) = -0.99$, $B(x_1) = -0.5$, and $\Phi(x_1) = 0$. One can see in the figure that, again, after the initial excitation stage, the system settles (beyond $x \approx -500$) in the spatially autoresonant regime, in which the phase mismatch oscillates around its slowly varying average autoresonant value. Nonetheless, Figs. 1 and 2 differ by a considerable depletion of the pump wave during the interaction, so, for example, at $x = 0$ the self-consistent theory predicts a $\sim 50\%$ reduction of κ^a as compared to the decoupled pump wave case. The autoresonance continues beyond the $x = 0$ point, until the amplitude of the pump wave becomes so small that a phase detrapping process (opposite to the phase trapping stage in the initial interaction region) takes place and the roles of the pump and daughter waves are interchanged. In this detrapping stage, the pump wave com-

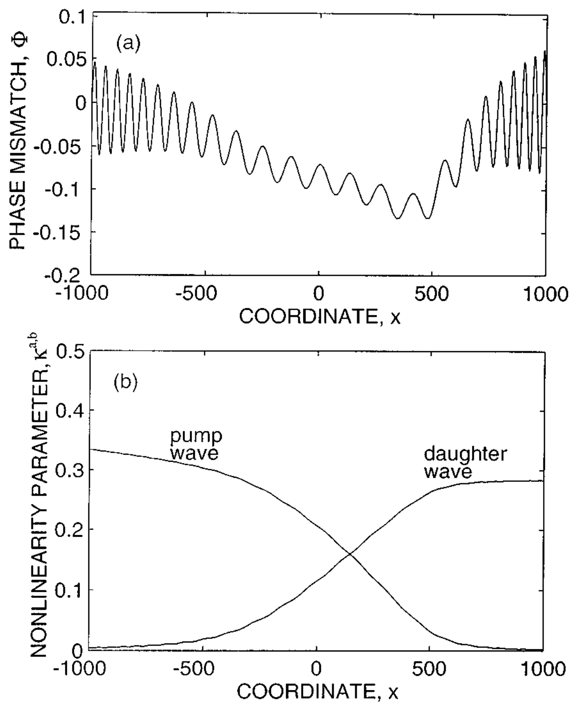

FIG. 2. Self-consistent, spatially autoresonant evolution of the solutions of linearly coupled sine-Gordon equations. (a) The phase mismatch vs x. (b) The nonlinearity parameters of the daughter and pump waves vs x. Three distinct stages of interaction in the figure are the initial excitation stage ($x < -500$), the autoresonant stage ($-500 < x < 500$), and the phase detrapping and saturation stages ($x > 500$).

prises a small amplitude (linear) wave driven by a large amplitude daughter wave. One can see, in Fig. 2, that the system enters the detrapping stage at $x \approx +500$ and the growth of the daughter wave saturates shortly beyond this point.

V. CONCLUSIONS

We have studied the problem of autoresonance of weakly coupled nonlinear waves with adiabatically space-time varying parameters. The autoresonant solutions for the interacting waves comprise two coexisting locally quasiuniform wave train solutions for formally decoupled daughter and pump waves, which at the same time are globally phase locked in an extended region of space-time.

The present work is a generalization of the previous theory [9] of the autoresonant evolution of nonlinear waves driven by a *prescribed* pump wave, and it also uses the averaged variational principle. The new ingredients in this study are the self-consistent inclusion of the autoresonant evolution of the pump wave, the investigation of the stability of the complete daughter-pump wave system, and the demonstration of the autoresonance in the case of weakly coupled sine-Gordon equations.

We have shown that the autoresonant interaction of coupled nonlinear waves involves three stages. The first stage proceeds at the boundary of the region of interest, where one launches a quasiuniform pump wave train toward the region where it resonates with an initially linear daughter wave (typically this region is a three-dimensional surface in four-dimensional space-time). This is the *initial excitation* stage, where the daughter wave is small and the pump wave can be treated as propagating independently in the adiabatically varying medium. At the linear resonance surface the daughter wave is excited and, under certain conditions, the system enters the *autoresonant interaction* stage. Here the two waves are globally phase locked and automatically adjust their amplitudes to preserve the nonlinear resonance condition. The autoresonance needs both the adiabatically and a sufficient nonlinearity of the coupled waves [see inequalities (36) and (39)]. Finally, as the energy density of the autoresonant daughter wave increases, the pump wave may be strongly depleted and the *phase detrapping* process takes place. In this, phase detrapping stage the roles of the pump and the daughter waves are interchanged, the pump wave gradually vanishes, and the growth of the daughter wave saturates as it propagates independently in the medium beyond the phase detrapping region.

We have illustrated our theory by examples of the spatial autoresonance in a driven sine-Gordon equation case and in the system of coupled sine-Gordon equations. The predictions of our adiabatic theory in the former example were found in an excellent agreement with the results of the direct numerical solutions, providing an indirect test of the applicability of the adiabatic theory in the self-consistent problem of autoresonance of coupled nonlinear waves.

The present theory is general in the sense that it is applicable to resonant interactions of weakly coupled nonlinear waves described by the variational principle with adiabatically varying parameters. Nonetheless, Lagrangian (5) considered in this study was restricted to waves described by second order PDE's. We plan to generalize this theory to study the autoresonant excitation and control of nonlinear waves described by higher order differential equations.

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